Gravitational Radiation from Primordial Helical Turbulence

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In present talk I show that primordial helical turbulence produced during a first-order phase transition induces circularly polarized cosmological gravitational waves (GWs). The degree of polarization as well as the characteristic frequency and the amplitude of these GWs depend crucially on phase-transition model of primordial turbulence. I present a brief discussion on the possibility of detection.

1. Introduction

There are several astrophysical observations indicating that the magnetic field in the Sun, and some galaxies and clusters of galaxies might have an helical structure [1]. On the other hand, recently it has been realized that magnetic (or/and kinetic) helicity can be generated during an early epoch of the universe [2,3]. Primordial helical motions influence magneto-hydrodynamical processes in the early plasma as well as cosmological perturbation dynamics [4,5]. Since cosmological gravitational waves (GWs) propagate without significant interaction after they are produced, once detected they should provide a powerful tool for studying the early Universe at the time of GW generation [6]. For the case at hand, GWs are considered as a test for primordial helical turbulence. The present formalism is general and can be applied to study the generation of stochastic GWs by any helical vector field (e.g., helical magnetic fields [7,8,9]).

It should be noted that various mechanisms for GW generation have been studied, including: quantum fluctuations during inflation [10]; bubble wall motion and collisions during phase transitions [11]; cosmological magnetic fields [12,13]; and plasma turbulence [13,14,15]. Here I focus on gravitational radiation generated by primordial helical turbulence [2,9]. This talk is based on results obtained in collaboration with G. Gogoberidze, A. Kosowsky, A. Mack, and B. Ratra (see Refs. [13,16]). We find that helical turbulence generates circularly polarized stochastic gravitational waves (GWs) and we compute the polarization degree. Primordial polarized GWs might be generated from quantum fluctuations accounting for the gravitational Chern-Simons term [17].

As it is well known GWs are generated by the transverse and traceless part of the stress-energy tensor $T_{\mu\nu}$ [18]. In our case $T_{\mu\nu}$ describes a turbulent cos-

mological fluid after a phase transition [11,13,14,15]. For spatial indices $i \neq j$, $T_{ij}(\mathbf{x}) = (p+\rho)u_i(\mathbf{x})u_j(\mathbf{x})$, where p and ρ are the fluid pressure and energy density and $\mathbf{u}(\mathbf{x})$ is the fluid velocity. The fluid enthalpy density $p+\rho$ is taken to be constant throughout space. The transverse and traceless part of T_{ij} in Fourier space is $\Pi_{ij}(\mathbf{k},t) = [P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}})]T_{lm}(\mathbf{k},t)$, where the projector onto the transverse plane $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ with $\hat{k}_i = k_i/k$, where k_i is the wave-vector. Consistent with observations we have assumed flat space sections.

Since turbulence has a stochastic nature the induced GWs should be also stochastic. They will form relic GWs background, and if there is a parity breaking in the source, one can expect the circular polarization of such waves [16]. To compute the gravitational waves signal from helical turbulence, first the model of turbulence should be specified.

2. Model helical turbulence

To model the turbulence we assume that in the early Universe at time $t_{\rm in}$ (at a phase transition) liberated vacuum energy $\rho_{\rm vac}$ is converted into (turbulent) kinetic energy of the cosmological plasma with an efficiency κ over a time scale $\tau_{\rm stir}$ on a characteristic source length scale L_S [11]. After generation, the turbulence kinetic energy cascades from larger to smaller scales. The cascade stops at a damping scale, L_D, where the turbulence energy is removed by dissipation. As usual, we assume that the turbulence is produced in a time much less than the Hubble time, $\tau_{\rm stir} \ll 1/H_{\rm in}$ — here $H_{\rm in}$ is the Hubble parameter at $t_{\rm in}$ [13,14], and rapidly generates GWs. We therefore ignore the expansion of the Universe when studying the generation of GWs (see below).

To compute the induced GW power spectrum one must have the source two-point function $\langle \Pi^\star_{ij}(\mathbf{k},t)\Pi_{lm}(\mathbf{k}',t')\rangle$. This is determined by the fluid velocity two-point correlation function. For stationary, isotropic and homogeneous flow the velocity two-point function is [7,9]

$$\langle u_i^\star(\mathbf{k})u_j(\mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')[P_{ij}P_S(k) + i\epsilon_{ijl}\hat{k}_l P_H(k)]. \tag{1}$$

Here $P_S(k)$ and $P_H(k)$ are the symmetric (related to the kinetic energy density per unit enthalpy of the fluid) and helical (related to the average kinetic helicity $\langle \mathbf{u}\cdot(\nabla\times\mathbf{u})\rangle$) parts of the velocity power spectrum [7,9]. Causality requires $P_S(k) \geq |P_H(k)|$ (Schwartz inequality, see p. 161 of Ref. [19], and Refs. [20,21,2,3,4]).

However, in the present case, the source of turbulence acts for only a short time τ, possibly not exceeding the large-scale eddy turnover time τ_S (corresponding to length scale L_S) — for self-consistency, however, we assume that the source is active over a time $\tau = \max(\tau_{\rm stir}, \tau_S)$ [13] — resulting in a time-dependent velocity spectrum. To model the development of helical turbulence during the time interval $(t_{\rm in}, t_{\rm fi} = t_{\rm in} + \tau)$ we make several simplifying assumptions:

(a) Turbulent fluid kinetic energy is present on all scales in the inertial (Kolmogorov) range $k_S < k < k_D$. Here $k_S = 2\pi/L_S$ and $k_D = 2\pi/L_D$, and the inertial

range includes length scales $L \in (L_D, L_S)$, i.e., length scales shorter than those on which energy flows into the turbulence and larger than those on which the turbulence energy is dissipated. We also assume that the energy is injected into the turbulence continuously over a time τ, rather than as an instantaneous impulse [13,14,15,16].

(b) Unequal time correlations are modeled as [13,16]

$$\langle u_i^\star(\mathbf{k},t) u_j(\mathbf{k}',t')\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}')[P_{ij} F_S(k,t-t') + i\epsilon_{ijl}\hat{k}_l F_H(k,t-t')], \quad (2)$$

where the $t-t'$ dependence of the functions F_S and F_H reflects the assumption of time translation invariance. Since energy is injected continuously, at $t = t' \in (t_{\rm in}, t_{\rm fi})$, $F_S(k,0) = P_S(k)$ and $F_H(k,0) = P_H(k)$.

(c) The decay of non-helical turbulence is determined by a monotonically decreasing function $D_1(t)$ and $F_S(k,t) = P_S(k)D_1(t)$, p. 259 of Ref. [22]. Extending this assumption to the helical turbulence case we also model $F_H(k,t) = P_H(k)D_2(t)$, where $D_2(t)$ is another monotonically decreasing function. Since in the considered model most of the power is in the inertial range [16], for simplicity we discard power outside the inertial range by truncating P_S and P_H at $k < k_S$ and $k > k_D$, i.e., we assume P_S and P_H (and so F_S and F_H) vanish outside the inertial range.

(d) We model the power spectra by power laws, $P_S(k) \propto k^{n_S}$ and $P_H(k) \propto k^{n_H}$. For non-helical hydrodynamical turbulence the Kolmogorov spectrum has $n_S = -11/3$. It has been speculated that in a magnetized medium an Iroshnikov-Kraichnan spectrum with $n_S = -7/2$ might develop instead. The presence of hydrodynamical helicity makes the situation more complex. Two possibilities have been discussed. First, with a forward cascade (from large to small scales) of both energy and helicity (dominated by energy dissipation on small scales) one has spectral indices $n_S = -11/3$ and $n_H = -14/3$ (the helical Kolmogorov (HK) spectrum), p. 243 of Ref. [19]. Second, if helicity transfer and small-scale helicity dissipation dominate, $n_S = n_H = -13/3$ (the helicity transfer (HT) spectrum) [21]. The HK spectrum has been observed in the inertial range of weakly helical turbulence (i.e., where $|P_H(k)| \ll P_S(k)$) [23]. For strongly helical hydrodynamical turbulence the characteristic small-scale length scale of helicity dissipation is larger than the Kolmogorov energy dissipation length scale [24]. Therefore here the inertial range is taken to consist of two sub-intervals, both with power-law spectra. For smaller k the spectra are determined by helicity transfer and have $n_S = n_H = -13/3$, while for larger k turbulence becomes non-helical and the more common HK spectrum is realized. Since GW generation is mostly determined by the physics at small k (near k_S) [13,14], it is fair to only use the HT spectrum in this case also [16].

Based on these considerations we model the primordial spectra as $P_S(k) = S_0 k^{n_S}$ and $P_H(k) = A_0 k_S^{n_S - n_H} k^{n_H}$, where: (i) for the HK case $S_0 = \pi^2 C_k \bar{\varepsilon}^{2/3}$ and $A_0 = \pi^2 C_k \bar{\delta}/(\bar{\varepsilon}^{1/3} k_S)$ [24], implying $A_0/S_0 = \bar{\delta}/(\bar{\varepsilon} k_S)$; and, (ii) for the HT case $S_0 = \pi^2 C_s \bar{\delta}^{2/3}$ and $A_0 = \pi^2 C_a \bar{\delta}^{2/3}$ [21]. Here $\bar{\varepsilon}$ and $\bar{\delta}$ are the energy and mean helicity dissipation rates per unit enthalpy, and C_k (the Kolmogorov constant), C_s, and C_a are constants of order unity.

The formalism described here is applicable for locally isotropic turbulence. Another requirement is connected with plasma viscosity, i.e. the Reynolds number Re, which is related to the cut-off scales k_S and k_D (for HK case, $\mathrm{Re} = (k_D/k_S)^{4/3}$), has to be enough large [25]. Summarizing, to model cosmological turbulence it is required to know [16]: (i) the part of the vacuum energy converted to the turbulent motions $\kappa\rho_{\rm vac}$; (ii) the "stirring" scale L_S; (iii) the temperature of the universe $T_{\rm in}$ when turbulence is generated. This temperature defines the energy density and enthalpy $(\rho + p)$ of the plasma; (iv) the ratio between energy and helicity dissipation rates $\bar{\delta}/(k_S\bar{\varepsilon})$. Knowing these quantities and the turbulent motion spectral index, we are able to obtain the damping scale k_D and the plasma viscosity ν (p. 483 of Ref. [25]). In particular, for locally isotropic turbulence the energy dissipation rate $\bar{\varepsilon} = 2\nu \int_{k_S}^{k_D} dk\; k^4 P_S(k)/\pi^2$ is equal to the source power input, i.e.,

$$\frac{4}{3}\bar{\varepsilon} = \frac{\kappa\rho_{\rm vac}}{\rho\tau}, \tag{3}$$

where κ is the phase transition efficiency. For HK turbulence with $n_S = -11/3$, the damping scale is related to plasma viscosity via [13]

$$\frac{9}{2}\nu^3\tau k_D^4 = \frac{\kappa\rho_{\rm vac}}{\rho}. \tag{4}$$

For the HT spectrum ($n_S = -13/3$) the damping scale is connected with plasma viscosity and the dissipation rates $\bar{\varepsilon}$ and $\bar{\delta}$ via $3\nu(\bar{\delta}k_D)^{2/3} = \bar{\varepsilon}$; using the fact that the total energy density of turbulent motion $\rho_{\rm turb} = \kappa\rho_{\rm vac}$, we find [16]

$$\kappa\left(\frac{k_S}{k_D}\right)^{2/3} = 3\tau\nu k_S^2 \tag{5}$$

The L-scale eddy turnover (circulation) time can be obtained as a ratio of length scale L to the physical velocity v_L related to the normalization of the symmetric part of turbulent motion spectrum $P_S(k)$ (or the energy dissipation rate $\bar{\varepsilon}$). It is easy to find that $2\tau_L \simeq 3\bar{\varepsilon}^{-1/3}L^{2/3}$ for any L inside the inertial range $L_D < L < L_S$ [13,16].

It should be underlined that we use the non-relativistic turbulence model to describe processes in the early universe. The Kolmogorov model of turbulence has been proved only for non-relativistic velocities in plasma, and there is no standard model for the relativistic case. We have also used the non-compressible plasma approximation. For the realistic case of a compressible fluid one can expect that a large enough input of energy may lead to the formation of shocks. On the other hand, our assumption that the upper limit of plasma velocity is equal to sound speed may be justified because of significant thermal dissipation due to shock waves. Presuming that shock fronts accumulate kinetic energy, the gravitational radiation signal will be increased (relative to the non-relativistic case), if we account for relativistic effects [13,14].

3. gravitational wave polarization degree

Since the turbulent source acts a short time (comparing with Hubble time-scale), the expansion of the universe can be neglected in the GW equation of motion, then in wave-number space we get [18]

$$\ddot{h}_{ij}(\mathbf{k},t) + k^2 h_{ij}(\mathbf{k},t) = 16\pi G \Pi_{ij}(\mathbf{k},t). \tag{6}$$

Here G is the Newtonian gravitational constant. We use natural units $\hbar = 1 = c$, physical/proper wave-numbers (not co-moving ones), and an over-dot denotes a derivative with respect to time t. $h_{ij}(\mathbf{k},t) = \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}} h_{ij}(\mathbf{x},t)$ and $h_{ij}(\mathbf{x},t) = \int d^3k\, e^{-i\mathbf{k}\cdot\mathbf{x}} h_{ij}(\mathbf{k},t)/(2\pi)^3$ is the Fourier transform pair of the transverse-traceless tensor metric perturbation which is defined in the terms of the complete metric perturbation $h_{ij} = \delta g_{ij}$ subject to the conditions $h_{ii} = 0$ and $h_{ij}\hat{k}^j = 0$. Choosing the coordinate system so that unit vector $\hat{\mathbf{e}}_3$ points in the GW propagation direction, using the usual circular polarization basis tensors $e^{\pm}_{ij} = -(\mathbf{e}_1 \pm i\mathbf{e}_2)_i \times (\mathbf{e}_1 \pm i\mathbf{e}_2)_j/\sqrt{2}$ [18], and defining two states h^+ and h^- corresponding to right- and left-handed circularly polarized GWs, we have $h_{ij} = h^+ e^+_{ij} + h^- e^-_{ij}$.

Stochastic turbulent fluctuations generate stochastic GWs. Gaussian-distributed GWs may be characterized by the wavenumber-space two-point function

$$\langle h^\star_{ij}(\mathbf{k},t) h_{lm}(\mathbf{k}',t)\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') \left[\mathcal{M}_{ijlm} H(k,t) + i\mathcal{A}_{ijlm}\mathcal{H}(k,t)\right]. \tag{7}$$

Here $H(k,t)$ and $\mathcal{H}(k,t)$ characterize the GW amplitude and polarization, $4\mathcal{M}_{ijlm}(\hat{\mathbf{k}}) \equiv P_{il}P_{jm} + P_{im}P_{jl} - P_{ij}P_{lm}$, and $8\mathcal{A}_{ijlm}(\hat{\mathbf{k}}) \equiv \hat{\mathbf{k}}_q(P_{jm}\epsilon_{ilq} + P_{il}\epsilon_{jmq} + P_{im}\epsilon_{jlq} + P_{jl}\epsilon_{imq})$ are tensors, and ϵ_{ijl} is the 3 dimentional fully antisymmetric symbol.

The GW degree of circular polarization is given by [26]

$$\mathcal{P}(k,t) = \frac{\langle h^{+\star}(\mathbf{k},t)h^+(\mathbf{k}',t) - h^{-\star}(\mathbf{k},t)h^-(\mathbf{k}',t)\rangle}{\langle h^{+\star}(\mathbf{k},t)h^+(\mathbf{k}',t) + h^{-\star}(\mathbf{k},t)h^-(\mathbf{k}',t)\rangle} = \frac{\mathcal{H}(k,t)}{H(k,t)}. \tag{8}$$

Both $H(k,t)$ and $\mathcal{H}(k,t)$ are obtained by solving Eq. (6), and are related to $\Pi_{ij}(\mathbf{k},t)$. For instance, an axisymmetric stochastic vector source (non-helical turbulent motion or any other non-helical vector field) induces unpolarized GWs with $|h^+(\mathbf{k},t)| = |h^-(\mathbf{k},t)|$ [12,13,14]. On the other hand, the presence of a helical source (i.e., a source that breaks parity symmetry) alters this situation [16].

Given a model of the turbulence, the turbulent source two-point function is [16]

$$\langle \Pi^\star_{ij}(\mathbf{k},t)\Pi_{lm}(\mathbf{k}',t+y)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\left[\mathcal{M}_{ijlm} f(k,y) + i\mathcal{A}_{ijlm} g(k,y)\right], \tag{9}$$

where $\mathcal{M}_{ijlm}$ and $\mathcal{A}_{ijlm}$ are defined below Eq. (7). The functions $f(k,y)$ and $g(k,y)$ that describe the symmetric and helical parts of the two-point function are [16]

$$f(k,y) = \frac{(\rho+p)^2}{256\pi^6}\int d^3p_1\int d^3p_2\delta^{(3)}(\mathbf{k}-\mathbf{p_1}-\mathbf{p_2})\,[(1+\gamma^2)(1+ \\ +\beta^2)D_1^2(y)P_S(p_1)P_S(p_2)+4\gamma\beta D_2^2(y)\,]\,P_H(p_1)P_H(p_2), \tag{10}$$

$$g(k,y) = \frac{(\rho+p)^2 D_1(y)D_2(y)}{128\pi^6}\int d^3p_1\int d^3p_2\,\delta^{(3)}(\mathbf{k}-\mathbf{p_1}-\mathbf{p_2})\,[(1+ \\ +\gamma^2)\beta P_S(p_1)P_H(p_2)(1+\beta^2)\gamma P_H(p_1)P_S(p_2)\,]\,, \tag{11}$$

where $\gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_1$ and $\beta = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_2$. The antisymmetric (parity-odd) source term $g(k,y)$ vanishes for turbulence without helicity.

To determine $H(k,t)$ and $\mathcal{H}(k,t)$ we solve Eq. (6) assuming that there is no GW for times $t<t_{\rm in}$, i.e., we choose as initial conditions $h_{ij}(\mathbf{k},t_{\rm in}) = 0 = \dot{h}_{ij}(\mathbf{k},t_{\rm in})$. To compute the induced GW power spectrum we use the averaging technique described in detail in Sec. III.B of Ref. [13]. The main points are: (i) the $\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ in Eqs. (7) and (9) ensure statistical isotropy of the GWs and allow $\mathbf{k}'$ and $\mathbf{k}$ to be interchanged; (ii) the statistical average can be approximated by either a time or a space average (this is justified for locally isotropic turbulence where the correlation function $\langle u_i(\mathbf{x}_0,t_0)u_j(\mathbf{x}_0+\mathbf{r},t_0+t)\rangle$ for $t\le\tau$ does not depend on t_0 for relatively small t, see Sec. 21.2 of Ref. [25]); (iii) we choose to time average since the Green function for Eq. (6) and the source term $\Pi_{lm}(\mathbf{k},t)$ are time dependent (we then have three time integrals for the GW two-point function, two from the GW Green function solutions and one from the time averaging, $\int_{t_1}^{t_1+T}dt/T$, where t_1 is an arbitrary time during the time interval τ when the source is active and the averaging time $T\le\tau$); (iv) we take $Tk\gg 1$ (since k^{-1} is of order the light crossing time for scale $L\sim k^{-1}$ while T is of order the L scale eddy turnover time). These approximations result in Eq. (32) of Ref. [13],

$$\langle h^\star_{ij}(\mathbf{k},t_{\rm fi})h_{lm}(\mathbf{k}',t_{\rm fi})\rangle \simeq \frac{(16\pi G)^2\tau}{2kk'}\int_{t_{\rm in}}^{t_{\rm fi}} dt\;\cos(kt)\langle\Pi^\star_{ij}(\mathbf{k},t_1)\Pi_{lm}(\mathbf{k}',t_1+t)\rangle. \tag{12}$$

The two-point function on the r.h.s. of the integral is independent of t_1 and $\langle h^\star_{ij}(\mathbf{k},t_{\rm fi})h_{lm}(\mathbf{k}',t_{\rm fi})\rangle$ is proportional to the source duration time τ, as expected for locally isotropic turbulence (for a more detailed discussion see p. 358 of Ref. [25]).

From Eqs. (9)–(12) we see that the symmetric $H(k,t)$ and helical $\mathcal{H}(k,t)$ parts of the GW two-point function in Eq. (7) are integrals over y of $\cos(ky)D_1^2(y)$, $\cos(ky)D_2^2(y)$, and $\cos(ky)D_1(y)D_2(y)$. Both $D_1(y)$ and $D_2(y)$ are positive monotonically-decreasing functions of y, and since $\cos(ky)$ oscillates on a time scale shorter than the characteristic decay time for the $D(y)$'s, the integrands oscillate and $\int_{t_{\rm in}}^{t_{\rm fi}}dy\cos(ky)F_a(p,y)F_b(|\mathbf{k}-\mathbf{p}|,y)\simeq P_a(p)P_b(|\mathbf{k}-\mathbf{p}|)/(\sqrt{2}k)$ (where a and b

can be S or H). Integrating over angles, we find at $t = t_{\rm fi}$ [16],

$$H(k) \simeq A \int dp_1\ p_1 \int dp_2\ p_2 \bar{\Theta} \left[(1+\gamma_p^2)(1+\beta_p^2) P_S(p_1) P_S(p_2) + \right.$$
$$\left. + 4\gamma_p \beta_p P_H(p_1) P_H(p_2) \ \right], \tag{13}$$
$$\mathcal{H}(k) \simeq 2A \int dp_1\ p_1 \int dp_2\ p_2 \bar{\Theta} \left[(1+\gamma_p^2)\beta_p P_S(p_1) P_H(p_2) + \right.$$
$$\left. + (1+\beta_p^2)\gamma_p P_H(p_1) P_S(p_2) \ \right]. \tag{14}$$

Here $A = \alpha\tau/(4\pi^2 k^4)$ where $\alpha = \sqrt{2}(p+\rho)^2(8\pi G)^2$, $\gamma_p = (k^2+p_1^2-p_2^2)/(2kp_1)$, $\beta_p = (k^2+p_2^2-p_1^2)/(2kp_2)$, $\bar{\Theta} \equiv \theta(p_1+p_2-k)\theta(p_1+k-p_2)\theta(p_2+k-p_1)$, and θ is the Heaviside step function which is zero (unity) for negative (positive) argument. Let's note that present approximations differs from ones used previously in Refs. [13,8]. The difference comes from using $\delta^{(3)}(\mathbf{k}-\mathbf{p})$ function to evaluate angle integrals and don't absorb modulus integration $|\mathbf{p}|$. This preserves to make errors when approaching the integration edges. In particular , since the spectra $P(|\mathbf{k}-\mathbf{p}|)$ are defined only for $k_S < |\mathbf{k}-\mathbf{p}| < k_D$, then the integration over spatial angle γ can not range from -1 to 1 as it is usually done making the convolution when the spectra are defined in whole of $p \in (0, \infty)$.

For power-law $P_S(k) \propto k^{n_S}$ and $P_H(k) \propto k^{n_H}$ the integrals in Eqs. (13) and (14) can be done analytically, but the results are complicated and do not edify. Instead we compute the degree of circular polarization, Eq. (8), by evaluating the integrals numerically for different parameter values [16]. Results are shown in Fig. 1.

Figure 1 and other numerical results show that for maximal helicity turbulence (when $A_0 = S_0$) with equal spectral indices $n_H = n_S < -3$, the polarization degree $\mathcal{P}(k) \simeq 1$ (upper solid line). For weaker helical turbulence (when $A_0 < S_0$) with $n_H \simeq n_S < -3$, $\mathcal{P}(k) \to CA_0/S_0$, where $1 < C(n_S, n_H) < 2$ is a numerical factor that depends on the spectral indices. For HT turbulence with $n_S = n_H = -13/3$, $C \approx 1.50$, while for Iroshnikov-Kraichnan MHD turbulence ($n_S = n_H = -7/2$), $C \approx 1.39$. Excluding the edges of the inertial range $k_S < k < k_D$, an analysis of Eqs. (13) and (14) shows that the main contribution to the integrals come from areas with $p_1 \sim k_S,\ p_2 \sim k$ and $p_1 \sim k,\ p_2 \sim k_S$. In this case (for arbitrary spectral indices $n_S,\, n_H < -3$) $H(k),\ \mathcal{H}(k) \propto k^{n_S-3} k_S^{n_S+3}$,

4. Results and discussion

To make the connection with the GW measurements, we have to define the real-space two point correlation function $\langle |h^{\pm}(\mathbf{x}, t_{\rm end})|^2 \rangle$. We find

$$\langle |h^{\pm}(\mathbf{x}, t_{\rm fi})|^2 \rangle \simeq -\frac{9\alpha\tau S_0^2 k_S^{n_S+3}}{64\pi^4(n_S+3)} \int_{k_S}^{k_D} dk (1 \pm \mathcal{P}(k)) k^{n_S-1}. \tag{15}$$

GW amplitudes are conventionally expressed as $\langle h^{ij}(\mathbf{x}, t_{\rm fi}) h^{ij}(\mathbf{x}, t_{\rm fi}) \rangle = 2\int_0^\infty d\ln f [|h^+(f)|^2 + |h^-(f)|^2]$, see Eq. (11) of Ref. [27], where the frequency f of

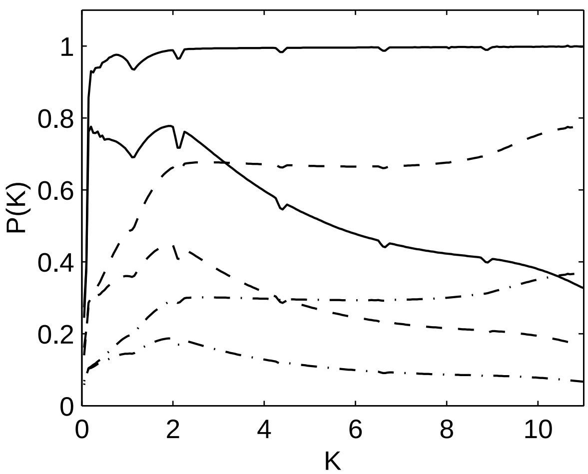

Fig. 1. GW polarization degree $\mathcal{P}(K,t_{\rm fi})$, Eq. (8), as a function of scaled wave-number $K = k/k_S$ relative to the large-scale wave-number k_S on which energy is pumped into the turbulence. This is evaluated at time $t_{\rm fi}$, after the turbulence has switched off, and remains unchanged to the present epoch. It has been computed for a damping wave-number $k_D = 10k_S$. Three pairs of curves are shown. Solid lines correspond to the amplitude ratio $A_0/S_0 = 1$, dashed lines to $A_0/S_0 = 0.5$, and dot-dashed lines are for $A_0/S_0 = 0.2$. The upper line in each pair corresponds to HT turbulence with $n_S = n_H = -13/3$ and the lower line to HK turbulence with $n_S = -11/3$ and $n_H = -14/3$. Even for helical turbulence with $A_0/S_0 \leq 0.5$, for large wave-numbers $k \sim k_D$, $n_S = n_H = -13/3$ is unlikely so the large K part of the lower dashed and dot-dashed HT curves are unrealistic. The large $k \sim k_D$ decay of the HK curves is a consequence of vanishing helicity transfer at large k

a GW generated by an eddy of length L is $f = \tau_L^{-1}$ where τ_L is the eddy turnover time [11,13,14]. Using $f = 2\bar{\varepsilon}^{1/3}L^{-2/3}/3$ we get

$$f \simeq \frac{1}{2\pi^2}\sqrt{\frac{S_0 k^{n_S+5}}{|n_S+3|}}, \qquad f_S \simeq 2L_S^{-\frac{n_S+5}{2}}\sqrt{\frac{(2\pi)^{n_S+1}S_0}{|n_S+3|}}, \tag{16}$$

where f_S is the frequency that corresponds to the stirring length L_S. Both the frequency and the amplitude of GWs are inversely proportional to the cosmological scale factor, so the frequency $\bar{f}$ today and f when the temperature was $T_{\rm in} = 100\ T_{100}$ GeV are related by $\bar{f} = 1.65 \times 10^{-5} T_{100} g_{100}^{1/6} f/H_{\rm in}$ Hz [27], where $g_{\rm in} = 100 g_{100}$ is the number of relativistic degrees of freedom at $t_{\rm in}$. Since we truncate turbulence power for $L > L_S$, the GW spectrum is non-zero only for $\bar{f} > \bar{f}_S$, where [16]

$$\bar{f}_S = 1.9 \times 10^{-6}\sqrt{\frac{n_S+5}{|n_S+3|}}\left(\frac{\bar{\varepsilon}}{\nu}\right)^{1/2}\left(\frac{L_D}{L_S}\right)^{(n_S+5)/2} T_{100}\ g_{100}^{1/6}\ H_{\rm in}^{-1}\ \text{Hz} \tag{17}$$

Here we used Eqs.(3)-(5). For HK turbulence ($n_S = -11/3$) the expression for $\bar{f}_S$ will be transformed in Eq. (53) of Ref. [13].

Using Eqs. (15) and (17), and neglecting the weak k-dependence of the GW polarization degree $\mathcal{P}$, we find that $h^{\pm}(\bar{f}) \propto \bar{f}^{-11/4}$ for the HK case [13,14], while for

HT turbulence $h^{\pm}(\bar{f}) \propto \bar{f}^{-13/2}$. We expect such a steeper dependence on frequency for helicity induced GWs, since the helicity transfer rate is more important on larger scales. In both cases the amplitude of the GW spectrum peaks at the stirring frequency $\bar{f}_S$.

To examine the prospect of such circularly polarized GWs detection, we compute the energy density parameters of GWs. This depends strongly on the frequency band. The GW energy density parameter for frequency $\bar{f}$, $\Omega_{\rm GW}(\bar{f})$ is given by (see Eq. (7) of Ref [27])

$$\Omega_{\rm GW}(\bar{f})h^2 = 5.9 \times 10^{35}(|h^+(\bar{f})|^2 + |h^-(\bar{f})|^2)(\frac{\bar{f}}{\rm Hz})^2, \tag{18}$$

where h is the Hubble constant in units of 100 km sec^{-1} Mpc^{-1}. In our case,

$$\Omega_{\rm GW}(\bar{f})h^2 \simeq 1.05 \times 10^{-11}\ g_{100}^{-1/3} \left(\frac{L_S^2}{\tau H_{\rm in}^{-1}} \cdot \frac{n_S+5}{|n_S+3|}\right)^2 \left(\frac{L_D}{L_S}\right)^{3(n_S+5)} \left(\frac{3\kappa\rho_{\rm vac}L_S}{4\nu\rho}\right)^3 \left(\frac{\bar{f}}{\bar{f}_S}\right)^{2(2n_S+5)/(n_S+5)} . \tag{19}$$

As it is expected the stirring frequency $\bar{f}_S$ and the GW spectrum are very sensitive to phase transition properties (for details see Fig. 1 of Ref. [14]). If the phase transition is strongly first order, $\bar{f}_S \simeq 5\times10^{-3}$Hz (for the HK case) [13], is near the LISA frequency range, but the amplitude of the GW signal is below LISA sensitivity [14,15,6]. Also the gravitational radiation signal coming from white dwarf binaries [28] will be compatible by amplitude with relic signal considered above (for the frequency range around $10^{-3} - 10^{-2}$ Hz making difficult to distinguish the source of gravitational waves. Thus it is unlikely that the circularly-polarized GWs considered here will be detected in the near future, however, GWs generated by helical turbulence will have a enough high degree of circular polarization and future detector configurations [29] may well be able to.

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